A TROPICAL APPROACH TO SECANT DIMENSIONS

JAN DRAISMA

Abstract

Tropical geometry yields good lower bounds, in terms of certain combinatorial-polyhedral optimisation problems, on the dimensions of secant varieties. In particular, it gives an attractive pictorial proof of the theorem of Hirschowitz that all Veronese embeddings of the projective plane except for the quadratic one and the quartic one are non-defective; this proof might be generalisable to cover all Veronese embeddings, whose secant dimensions are known from the ground-breaking but difficult work of Alexander and Hirschowitz. Also, the non-defectiveness of certain Segre embeddings is proved, which cannot be proved with the rook covering argument already known in the literature. Short self-contained introductions to secant varieties and the required tropical geometry are included.

Acknowledgments

I thank Tony Geramita, Rick Miranda, Bernd Sturmfels, and Seth Sullivant very much for their responses to an earlier version of this paper. In particular, they pointed out several very nice papers on secant varieties of which I was unaware. Also, their responses raise the question of how the tropical approach relates to the degeneration approach of Sturmfels-Sullivant and Dumitrescu-Miranda. This question and the combinatorial-polyhedral problems introduced below will certainly lead to further research. Finally, I thank Hannah Markwig for pointing out a gap in the proof of Theorem 4.2 in an earlier version of this paper, and Marius van der Put who pointed out a theorem about affinoid algebras which can be used to fill that gap.

1. Introduction

Secant varieties are rather classical objects of study in algebraic geometry: given a closed subvariety X of some projective space $\mathbb{P}^m$, and given a natural number k, one tries to describe the union of all subspaces of $\mathbb{P}^m$ that are spanned by k points on X. We call the Zariski closure of this union the k-th *secant variety* of X, and denote it by kX. To avoid confusion: some authors call this the $(k-1)$st secant variety. So in this paper $2X$ is the variety of secant lines, traditionally called *the* secant variety of X. We will refer to all kX as (higher) secant varieties, and to their dimensions as (higher) secant dimensions. The standard reference for secant varieties is [30].

Already the most basic of all questions about the secant varieties of X poses unexpected challenges, namely: what are their dimensions? This question is of particular interest when X is a *minimal orbit* in a representation space of a reductive group. These minimal orbits comprise Segre embeddings of products of projective

spaces, Plücker embeddings of Grassmannians, and Veronese embeddings of projective spaces; see Section 6. Among these instances, only the secant dimensions of the Veronese embeddings are completely known, from the ground-breaking work of Alexander and Hirschowitz [1, 2, 3, 18]. Secant dimensions of Segre powers of the projective line are almost entirely known [11].

This paper introduces a new approach to secant dimensions, based on tropical geometry. Tropical geometry is well known as a tool for transforming algebraic-geometric questions into polyhedral-combinatorial ones. Recommended references are [7, 16, 22, 24, 25, 26] and the references therein—however, all background in tropical geometry needed here is reviewed in Section 4.

In Sections 2 and 3 I present the tropical lower bounds on secant dimensions in terms of certain polyhedral optimisation problems. After a review of the necessary tropical geometry in Section 4 we prove the lower bounds in Section 5. In Section 6 I recall the notion of minimal orbits, and give two lower bounds on their secant dimensions. One of them is well known in special cases; for instance, it uses rook coverings for Segre varieties, and a variation of these for Grassmannians [10, 12, 15, 28]. The other seems to be good for Segre products of Veronese embeddings.

Then in Section 7 we apply the tropical lower bounds to Segre varieties, Veronese embeddings, and Grassmannians, and show that they are better than the bounds from Section 6. As an example, we reprove the theorem that all but two Veronese embeddings of the projective plane are non-defective; this was proved earlier by Hirschowitz [18] using his "Horace method" and by Miranda and Dumitrescu using degenerations (private communication). Also, I give a nice proof that the 6-fold Segre power of the projective line is non-defective; this cannot be proved using rook coverings alone, and is the first case not covered by [11]. Finally, Seth Sullivant and Bernd Sturmfels pointed out the paper [14] to me, in which tropical secant varieties of ordinary linear spaces are considered. The precise relation between Develin's tropical secant varieties and the tropicalisation of secant varieties is still unclear to me, though under reasonable conditions the former should be contained in the latter.

In conclusion, the tropical approach is conceptually very simple, but shows very promising results when tested on concrete examples. However, it also raises many intriguing combinatorial-polyhedral optimisation problems; I do not know of any efficient programs solving these.

2. Joins, secant varieties, and first results

Rather than projective varieties, we consider closed cones in affine spaces. So let K be an algebraically closed field of characteristic 0, let V be a finite-dimensional vector space over K, and let C, D be closed cones: Zariski-closed subsets of V that are closed under scalar multiplication. Then we define the *join* of C and D as follows:

$$C + D := \overline{\{c + d \mid c \in C, d \in D\}}.$$

Note that in taking the closure we ignore the subtle question of which elements of $C + D$ can actually be written as $c + d$ with $c \in C$ and $d \in D$; in this paper we are only interested in dimensions, and hence there is no harm in taking the closure. There is an obvious upper bound on the dimension of $C + D$, namely $\min\{\dim C + \dim D, \dim V\}$—indeed, the summation map $C \times D \to C + D$ is dominant. We call this upper bound the *expected dimension* of $C + D$. If $C + D$ has

strictly lower dimension than expected, then we call $C + D$ *defective*; otherwise, we call $C + D$ *non-defective*. The difference $\min\{\dim C + \dim D, \dim V\} - \dim(C + D)$ is called the *defect*.

Taking the join is an associative (and commutative) operation on closed cones in V, so given k closed cones $C_1, \ldots, C_k$, their join $C_1 + \ldots + C_k$ is well-defined. Again, we call this join defective or non-defective according as its dimension is smaller than or equal to $\min\{\dim V, \sum_i \dim C_i\}$.

In particular, taking all C_i equal to a single closed cone C we obtain kC, called the kth *secant variety* of C. The defect of kC, also called the k-th secant defect of C, is defined in a slightly different manner: it is the difference $\min\{\dim V, \dim C + \dim(k-1)C\} - \dim kC$; hence if $k \dim C < \dim V$, then the difference $k \dim C - \dim kC$ is the *sum* of all l-defects for $l \leq k$. We call kC defective if its defect is positive, and non-defective otherwise. Finally, we call C itself defective if and only if kC is defective for some $k \geq 0$, and we call the numbers $\dim kC,\ k \in \mathbb{N}$ the *secant dimensions* of C. The standard reference for joins and secant varieties is [30].

Typically, one considers a class of cones (e.g., the cones over Grassmannians), one knows a short explicit list of defective secant varieties of cones in this class, and wishes to prove that all other secant varieties of cones in this class are non-defective. One then needs *lower bounds* on secant dimensions that are in fact *equal* to the expected dimensions—so that one can conclude equality.

Our approach towards such lower bounds focuses on the following, special situation: suppose that $C_1, \ldots, C_k$ are closed cones in V, and single out a basis $e_1, \ldots, e_n$ of V. The method depends on this basis, but in our applications there will be natural bases to work with. Let $y_1, \ldots, y_n$ be the dual basis of V^*. Assume for simplicity that none of the C_i is contained in any coordinate hyperplane $\{y_b = 0\}$. Furthermore, suppose that for each i we have a finite-dimensional vector space V_i over K, again with a fixed basis $x = (x_1, \ldots, x_{m_i})$ of V_i^*, and a polynomial map $f_i : V_i \to V$ that maps V_i dominantly into C_i. In particular, every C_i is irreducible.

Write each f_i, relative to the bases of V_i and V, as a list $(f_{i,b})_{b=1}^n$ of polynomials $f_{i,b} \in K[x_1, \ldots, x_{m_i}]$; the fact that we use the same letter x to indicate coordinates on the distinct V_i will not lead to any confusion. For every $i = 1, \ldots, k$ and $b = 1, \ldots, n$, let $l_{i,b}$ be the piecewise linear function $\mathbb{R}^{m_i} \to \mathbb{R}$ defined as follows: Write

$$f_{i,b} = \sum_\alpha c_\alpha x^\alpha,$$

where the sum runs over all multi-indices $\alpha \in \mathbb{N}^{m_i}$ for which c_α is non-zero; we view these α as row vectors, i.e., we will tacitly regard $\mathbb{N}^{m_i}$ as a subset of $(\mathbb{R}^{m_i})^*$. Note that this is not an empty sum by the assumption that C_i not lie in $\{y_b = 0\}$. Then $l_{i,b}$ is defined by

$$l_{i,b}(v) := \min_\alpha \langle v, \alpha \rangle, \quad v \in \mathbb{R}^{m_i},$$

where α runs over the same domain, where v is regarded a column vector, and where $\langle ., . \rangle$ denotes the natural pairing between column vectors and row vectors. Thus $l_{i,b}$ is a piecewise linear function, whose slopes correspond to the monomials in $f_{i,b}$.

Theorem 2.1. *The dimension of* $C_1+\ldots+C_k$ *is at least the (topological) dimension of the polyhedral set*

$$Q := \left\{ \left(\min_{i=1,\ldots,k} l_{i,b}(v_i) \right)_{b=1,\ldots,n} \mid v_i \in \mathbb{R}^{m_i} \text{ for all } i = 1,\ldots,k \right\}$$

in $\mathbb{R}^n$.

As it stands, this theorem may not sound very appealing. For a more concrete reformulation we proceed as follows. For $v = (v_1,\ldots,v_k) \in \prod_{i=1}^k \mathbb{R}^{m_i}$, $b \in \{1,\ldots,n\}$, and $i \in \{1,\ldots,k\}$ we say that i *wins* e_b (or b) *at* v provided that

(1) $l_{i,b}(v_i) < l_{j,b}(v_j)$ for all $j \neq i$, and
(2) $l_{i,b}$ is differentiable (hence linear) near v_i.

If this is the case, then we denote by $d_{v_i} l_{i,b}$ the differential $\mathbb{R}^{m_i} \to \mathbb{R}$ of $l_{i,b}$ at v_i; note that this is given by a row vector of natural numbers. If, on the other hand, no i wins e_b at v, then we say that *there is a tie on* e_b *at* v.

Definition 2.2. For $v = (v_1,\ldots,v_k) \in \prod_{i=1}^k \mathbb{R}^{m_i}$ and $i = 1,\ldots,k$ set

$$W_i(v) := \{b \in \{1,\ldots,n\} \mid i \text{ wins } b \text{ at } v\},$$

and call $W_i(v)$ the *winning set* of i at v. Collect the corresponding differentials $\mathbb{R}^{m_i} \to \mathbb{R}$ in the set

$$D_i(v) := \{d_{v_i} l_{i,b} \mid b \in W_i(v)\},$$

called the set of *winning directions* of i at v.

As we shall see in Section 5, the dimension of Q in the theorem is equal to the maximum, over all v, of

$$\sum_{i=1}^k \dim_{\mathbb{R}} \langle D_i(v) \rangle_{\mathbb{R}}.$$

This leads to the following corollary.

Corollary 2.3. *The dimension of* $C_1+\ldots+C_k$ *is at least the maximum, taken over all* $v = (v_1,\ldots,v_k) \in \prod_{i=1}^k \mathbb{R}^{m_i}$, *of the sum*

$$\sum_{i=1}^k \dim_{\mathbb{R}} \langle D_i(v) \rangle_{\mathbb{R}}.$$

In particular, if there exists a v *such that the set of winning directions at* v *of each* i *spans a space of dimension* $\dim C_i$ *in* $(\mathbb{R}^{m_i})^*$, *then the join* $C_1+\ldots+C_k$ *is non-defective.*

This corollary suggests the following strategy for proving that $C_1+\ldots+C_k$ is non-defective: try and find a point v at which each i wins a fair share of the basis $e_1,\ldots,e_n$—where fair means that the linear forms on $\mathbb{R}^{m_i}$ by means of which i wins its share, span a space of dimension $\dim C_i$. In the following section we make this strategy more concrete for the case of secant varieties, by making explicit the optimisation problem that needs to be solved to get a good lower bound on $\dim kC$.

3. Secant dimensions and some optimisation problems

Suppose that we want to find lower bounds on the secant dimensions of a single closed cone $C \subseteq K^n$, which as before is the closure of the image of a polynomial map $f = (f_1, \ldots, f_n) : K^m \to K^n = V$. For $b = 1, \ldots, n$ let $A_b \subseteq \mathbb{N}^m$ be the set of all α for which the monomial x^α has a non-zero coefficient in f_b. For $v = (v_1, \ldots, v_k) \in (\mathbb{R}^m)^k$, $i = 1, \ldots, k$, and $b = 1, \ldots, n$ we see that i wins e_b at v if and only if there is an $\alpha \in A_b$ which has $\langle v_i, \alpha\rangle < \langle v_j, \beta\rangle$ for all $(j, \beta) \in \{1, \ldots, k\} \times A_b$ unequal to (i, α). The winning set $W_i(v)$ of i at v is the set of all b with this property, and the set $D_i(v)$ of winning directions is the set of all such minimising α as b runs over $W_i(v)$. Hence we are led to consider the following optimisation problem, in which we relax the, at this point somewhat unnatural, restriction that all A_b lie in $\mathbb{N}^m$.

Problem 3.1 (LinearPartition(A, k))**.** *Let $A = (A_1, \ldots, A_n)$ be a sequence of finite subsets of $(\mathbb{R}^m)^*$ and let $k \in \mathbb{N}$. For $v = (v_1, \ldots, v_k) \in (\mathbb{R}^m)^k$ and $i = 1, \ldots, k$ define*

$$D_i(v) := \bigcup_{b=1}^{n} \{\alpha \in A_b \mid \langle v_i, \alpha\rangle < \langle v_j, \beta\rangle \quad \text{for all } (j, \beta) \in \{1, \ldots, m\} \times A_b \text{ unequal to } (i, \alpha)\}.$$

Maximise $\sum_{i=1}^k \dim\langle D_i(v)\rangle_{\mathbb{R}}$ *over all $v \in (\mathbb{R}^m)^k$.*

Note that at a given v, every A_b only contributes to at most one $D_i(v)$. The following corollary is immediate from Corollary 2.3.

Corollary 3.2. *The dimension of kC is at least the optimum of* LinearPartition(A, k).

The bad news is: this lower bound need not be very good. The good news: we will see in Section 6 that the bound is not useless—in particular for minimal orbits in representations where all weight spaces are one-dimensional—and in Section 7 I give examples where the bound is very good. The following examples illustrate both facts.

Example 3.3 (Bernd Sturmfels)**.** Let $f : K^m \to K^n$ be a linear map whose matrix entries are all non-zero. Then all A_b are equal to $\{e_1^t, \ldots, e_m^t\}$, where e_j is the j-th standard basis vector of $\mathbb{R}^m$. Let $v = (v_1, \ldots, v_k) \in (\mathbb{R}^m)^k$. Now if i wins b at v, then i wins all other b as well, and the winning directions α are all equal to the same e_j^t. Hence the lower bound of Corollary 3.2 on the dimension of kC is 1 for all $k \geq 1$—not very good indeed.

Example 3.4. Let $V = K[t_1, t_2]_d$ be the space of homogeneous polynomials of degree d in t_1, t_2, and let C be the cone in V of pure powers $(x_1t_1 + x_2t_2)^d$, $x_1, x_2 \in K$. We prove that all secant varieties of C are non-defective [18]. As a parametrisation of C we take the map $f : K^2 \to V$, $(x_1, x_2) \mapsto (x_1t_1 + x_2t_2)^d$. We fix the basis $t_1^b t_2^{d-b}$, $b = 0, \ldots, d$ of V. The coefficient of $t_1^b t_2^{d-b}$ in $f(x_1, x_2)$ is $\binom{d}{b} x_1^b x_2^{d-b}$, so that A_b consists of the vector $(b, d-b)$ only. Hence Corollary 3.2 suggests that we compute the optimum of LinearPartition$(\{\{d-b\}\}_{b=0}^d, k)$.

Now suppose first that $2k \leq d + 1$. Then it is not hard to find a $v = (v_1, \ldots, v_k)$ such that $D_{i+1}(v)$ contains both $(2i, d-2i)$ and $(2i+1, d-2i-1)$, for all $i = 0, \ldots, k-1$: Draw $k-1$ lines $l_1, \ldots, l_{k-1}$ in $\mathbb{R}^2_{\geq 0}$, all through 0 and such that l_i

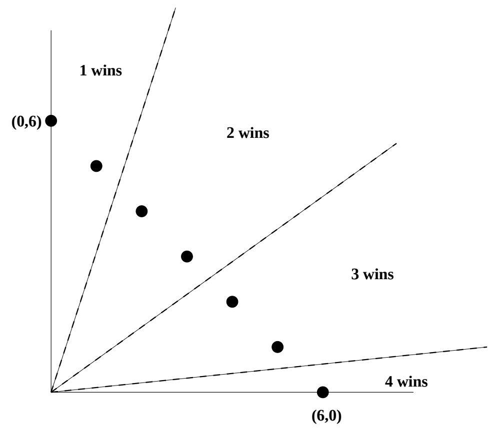


FIGURE 1. The secants of d-th pure powers in binary forms are non-defective.

separates $(2i, d-2i)$ from $(2i-1, d-2i+1)$; and imagine any concave piecewise linear (continuous) function F on $\mathbb{R}^2_{\geq 0}$ which is non-differentiable precisely in the l_i. Then the slope vectors v_i of F on the k components of the complement of the l_i do the job. So $D_{i+1}(v)$ contains two linearly independent vectors $(2i, d-2i), (2i+1, d-2i-1)$, hence span has dimension $2 = \dim C$. Using the corollary we conclude that kC is not defective.

If, on the other hand, $2k = d+2$, then one can find $v_1, \ldots, v_{k-1}$ with the property above, while k wins $(d, 0)$; see Figure 1. Using the corollary we conclude that $kC = V$.

Two variations of LINEARPARTITION will appear in the sequel. First, there is an affine version which is useful, for instance, when the map f is homogeneous, like in the preceding example. We use the notation $\mathrm{Aff}_{\mathbb{R}} D$ for the affine span of a subset D in a real vector space. By convention the dimension of $\emptyset = \mathrm{Aff}_{\mathbb{R}} \emptyset$ is -1.

Problem 3.5 (AFFINEPARTITION(A, k))**.** *Let $A = (A_1, \ldots, A_n)$ be a sequence of finite subsets of $(\mathbb{R}^m)^*$ and let $k \in \mathbb{N}$. For $v = (v_1, \ldots, v_k) \in (\mathbb{R}^m)^k$, $a = (a_1, \ldots, a_k) \in \mathbb{R}^k$, and $i = 1, \ldots, k$, set*

$$D_i(v, a) := \bigcup_{b=1}^{n} \{\alpha \in A_b \mid \langle v_i, \alpha \rangle + a_i < \langle v_j, \beta \rangle + a_j \text{ for all } (j, \beta) \in \{1, \ldots, m\} \times A_b \text{ unequal to } (i, \alpha)\}.$$

Maximise $\sum_{i=1}^{k} (1 + \dim \mathrm{Aff}_{\mathbb{R}} D_i(v))$ *over all* $(v, a) \in (\mathbb{R}^m)^k \times \mathbb{R}^k$.

Remark 3.6. The following obvious observation is sometimes useful: if $A = (A_1, \ldots, A_n)$ is as in AFFINEPARTITION, and $\pi : (\mathbb{R}^m)^* \to (\mathbb{R}^{m'})^*$ is an affine-linear map, then the optimum of AFFINEPARTITION(A, k) is at least that of AFFINEPARTITION$(\pi(A), k)$.

The third optimisation problem can be used when each component of the parametrisation f is a (non-zero multiple of a) single monomial, like in Example 3.4. This optimisation problem depends on the choice of a positive definite inner product $(.,.)$ on $\mathbb{R}^m$. We use this inner product to identify $(\mathbb{R}^m)^*$ with its dual $\mathbb{R}^m$, as well as to define a norm $||.||_2$ on $\mathbb{R}^m$.

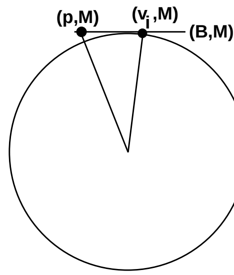


FIGURE 2. The distance in $B \times \{M\}$ is close to the spherical distance on a large sphere.

Problem 3.7 (VORONOIPARTITION(S,k))**.** *Let S be a finite subset of $\mathbb{R}^m$ and let $k \in \mathbb{N}$. For $v = (v_1, \ldots, v_k) \in (\mathbb{R}^m)^k$ let $\mathrm{Vor}_i(v)$ denote the intersection of S with the Voronoi cell of v_i, i.e.,*

$$\mathrm{Vor}_i(v) := \{\alpha \in S \mid ||v_i - \alpha||_2 < ||v_j - \alpha||_2 \textit{ for all } j \neq i\}.$$

Maximise $\sum_{i=1}^{k}(1 + \dim \mathrm{Aff}_{\mathbb{R}} \mathrm{Vor}_i(v))$ *over all $v \in (\mathbb{R}^m)^k$.*

The relations between these optimisation problems are as follows.

Lemma 3.8. *Let $A = (A_1, \ldots, A_n)$ be a sequence of finite subsets of $(\mathbb{R}^m)^*$, and let $k \in \mathbb{N}$.*

1. *If $\bigcup_b A_b$ is contained in an affine hyperplane not through the origin, then the optimum of* LINEARPARTITION(A,k) *equals the optimum of* AFFINEPARTITION(A,k).
2. *If every A_b is a singleton, then for $S := \bigcup_b A_b$ the optimum of* VORONOIPARTITION(S,k) *is a lower bound on the optimum of* AFFINEPARTITION(A,k).

Proof. For the first statement: the affine-linear functions on $W := \mathrm{Aff}_{\mathbb{R}} \bigcup_b A_b$ are precisely the restrictions to W of the linear functions on $\mathbb{R}^m$; and furthermore $\dim\langle M\rangle_{\mathbb{R}} = 1 + \dim \mathrm{Aff}_{\mathbb{R}} M$ for all $M \subseteq W$.

For the second statement, let $(v_1, \ldots, v_k)$ be an optimal solution to VORONOIPARTITION(S,k); I will argue that there exist $v' = (v'_1, \ldots, v'_k) \in (\mathbb{R}^m)^k$ and $a = (a_1, \ldots, a_k) \in \mathbb{R}^k$ such that

$$||v_i - \alpha||_2 < ||v_j - \alpha||_2 \Rightarrow (v'_i, \alpha) + a_i < (v'_j, \alpha) + a_j \tag{1}$$

for all $i, j = 1, \ldots, m$ and $\alpha \in S = \bigcup_b A_b$. This means that the $\mathrm{Vor}_i(v)$ for VORONOIPARTITION(A,k) are contained in the $D_i(v', a)$ for AFFINEPARTITION(A,k), whence the lemma follows.

Let B be a compact convex set in $\mathbb{R}^m$ containing the v_i and S; the latter is regarded as a subset of $\mathbb{R}^m$ through the inner product. Embed B in $\mathbb{R}^{m+1}$ by giving every point in B the same $(m+1)$-st coordinate $M > 0$; and extend the inner product to $\mathbb{R}^{m+1}$ by making e_{m+1} a norm-1 vector perpendicular to $\mathbb{R}^m$. By making M large B can be brought arbitrarily close to the sphere around the origin in $\mathbb{R}^{m+1}$ of radius M; see Figure 2. In particular, the function sending a point in B to its distance to v_i can be approximated, in the ∞-norm on continuous functions on B, arbitrarily well by the spherical distance

$$B \to \mathbb{R}, \ x \mapsto M \arccos \frac{((x,M),(v_i,M))}{||(x,M)||_2||(v_i,M)||_2}$$

This, in turn, implies that the intersection with B of the affine hyperplane with equation $||x - v_i||_2 = ||x - v_j||_2$ can be arbitrarily well approximated by the intersection with B of the affine hyperplane with equation

$$\frac{((x, M), (v_i, M))}{||(v_i, M)||_2} = \frac{((x, M), (v_j, M))}{||(v_j, M)||_2} \left(= (x, v_j/||(v_j, M)||_2) + M^2/||(v_j, M)||_2\right).$$

Hence, for v_i' we take $-v_i/||(v_i, M)||_2$, and for a_i we take $-M^2/||(v_i, M)||_2$; the minus signs ensure that the i-th affine-linear function is the *minimal* one near v_i, rather than the *maximal* one. Then, for M sufficiently large, (1) will be satisfied. $\square$

Note that application of VoronoiPartition makes the proof in Example 3.4 even easier: simply take v_i in the middle between $(2i-2, d-2i+2)$ and $(2i-1, d-2i+1)$ for $i = 1, \ldots, \lfloor \frac{d+1}{2} \rfloor$ and $v_{1+\frac{d}{2}}$ equal to $(d, 0)$ if d is even; and note that this v gives the maximal possible value for VoronoiPartition.

4. Tropical geometry

Tropical geometry turns questions about algebraic varieties into questions about polyhedral complexes, and this is precisely what the preceding sections do to secant dimensions. For the general set-up, let K be an algebraically closed field of characteristic 0, endowed with a non-archimedean valuation $v : K \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$, which may, and in our application will, be trivial. Let X be an affine algebraic variety over K and let $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)$ be an n-tuple of generators of $K[X]$, giving rise to a closed embedding $X \to K^n$. Let $x = (x_1, \ldots, x_n)$ be the standard coordinates on K^n. In general, too, we will write $\bar{f} \in K[X]$ for the restriction of a polynomial $f \in K[x]$ to X.

Definition 4.1. The *tropicalisation of X relative to $\bar{x}$* is

$$T_{\bar{x}}(X) := \left\{(v'(\bar{x}_1), \ldots, v'(\bar{x}_n)) \mid v' : K[X] \to \overline{\mathbb{R}} \text{ is a ring valuation extending } v\right\}.$$

Here a *ring valuation extending* v is a map $v' : K[X] \to \overline{\mathbb{R}}$ that equals v on K and satisfies the following axioms: $v'(0) = \infty$, $v'(fg) = v'(f) + v'(g)$ and $v'(f + g) \geq \min\{v'(f), v'(g)\}$ for all $f, g \in K[X]$. This definition, inspired by [7], is the cleanest definition of the tropicalisation of X. It shows clearly that $T_{\bar{x}}(X)$ is only a projection of an enormous object, namely the set of all ring valuations of $K[X]$ extending v. This fact explains why many things in tropical geometry go almost right, but not quite; and why tropicalisation does not have all the functorial properties one would like it to have. For instance, if X and Y are both closed subvarieties in K^n, and if we restrict the standard coordinates x on K^n to X, Y, and $X \cap Y$, respectively, retaining the notation $\bar{x}$ in all three cases, then we obtain three tropicalisations $T_{\bar{x}}(X)$, $T_{\bar{x}}(Y)$, $T_{\bar{x}}(X \cap Y)$ in $\overline{\mathbb{R}}^n$ and it is natural to ask: is $T_{\bar{x}}(X \cap Y) = T_{\bar{x}}(X) \cap T_{\bar{x}}(Y)$? While the inclusion $\subseteq$ is clear from the definition, the converse inclusion does not always hold. Another problem concerns morphisms: if $f : X \to Y$ is a morphism, and if $\bar{y}$ is a tuple of generators of $K[Y]$, then one would like to have a tropicalisation $T(f) : T_{\bar{x}}(X) \to T_{\bar{y}}(Y)$. Although there is a natural such map if all pullbacks $f^*(\bar{y}_j)$ are in the monoid generated by the $\bar{x}_i$ (and in particular $T(f)$ *can* be meaningfully defined on the tropical variety $T_{\bar{x}, f^*\bar{y}}(X)$) in general $T(f)$ cannot be defined on all of $T_{\bar{x}}(X)$ in a meaningful way. For first steps in the abstract theory of tropical varieties, see [23].

Turning to applications of tropical geometry, we need a more useful characterisation of $T_{\bar{x}}(X)$. What follows stays very close to the exposition in [26]. I include

it anyway for two reasons: First, for self-containedness, and second, because there are some slight differences: here we explicitly allow coordinates in X to become 0, hence to have valuation ∞; and we make a clear distinction between K and a rather large valued extension L that will soon play a role, thereby emphasising that tropical methods even apply when the original ground field is not endowed with a valuation.

For a $w \in \overline{\mathbb{R}}^n, c \in K$, and $\alpha \in \mathbb{N}^n$ we call $v(c) + \sum_i \alpha_i w_i$ the *w-weight* of the term $cx^\alpha \in K[x]$, written $\mathrm{wt}_w cx^\alpha$; here we extend $+$ to $\overline{\mathbb{R}}$ by $a + \infty = \infty$ for all $a \in \overline{\mathbb{R}}$ and set $0 \cdot \infty := 0$. For a polynomial $f \in K[x]$ we let $\mathrm{wt}_w f$ be the minimum of the weights of terms of f; in particular, $\mathrm{wt}_w 0 = \infty$. Define the *w-initial part* of f to be 0 if $\mathrm{wt}_w f = \infty$, and equal to the sum of all terms in f of weight $\mathrm{wt}_w f$ if the latter weight is $< \infty$.

Let (L, v) be an algebraically closed and complete extension of (K, v) with $v(L) = \overline{\mathbb{R}}$ (such an extension exists.) The definitions of wt_w and in_w extend naturally to L. The following theorem, which lies at the heart of tropical geometry, gives an alternative description of the tropicalisation $T_{\bar{x}}(X)$. Both [16] and [26] contain a version of this theorem; the proof below is close to that in the latter reference—except that, like in the first reference, some theory of affinoid algebras is used.

Theorem 4.2. *Let I be the ideal of X in $K[x]$. The following four sets are equal:*

1. $\{(v(\bar{x}_1(p)), \ldots, v(\bar{x}_n(p))) \mid p \in X(L)\}$,
2. $T_{\bar{x}}(X)$,
3. $\{w \in \overline{\mathbb{R}}^n \mid \mathrm{in}_w f$ *is not a monomial for any* $f \in I\}$, *and*
4. $\{w \in \overline{\mathbb{R}}^n \mid \mathrm{in}_w f$ *is not a monomial for any* $f \in L \otimes_K I\}$.

This is really remarkable: the set of all ring valuations of $K[X]$ extending v is huge and highly complicated. But this theorem says that when one is only interested in the values of those valuations on a *finite* tuple $\bar{x}$ of generators of $K[X]$, then one needs only consider the natural point valuations of $K[X]$ at L-rational points of X. The proof uses the following two lemmas, both of which need only slightly weaker assumptions on L or K.

Lemma 4.3. *Let L be a field with a non-Archimedean valuation v and let K be a subfield of L. Let A be an $r \times s$-matrix with entries in K, let $b \in K^r$, and let $l_1, \ldots, l_s$ be real numbers. Suppose that there exists a $y \in L^s$ for which*

$$v((Ay - b)_i) > l_i \text{ for all } i = 1, \ldots, r.$$

Then there also exists an $z \in K^s$ for which

$$v((Az - b)_i) > l_i \text{ for all } i = 1, \ldots, r.$$

First year linear algebra students know that if an *exact* solution to the system $Az = b$ exists over L, then also one exists over K. This lemma states that the same is true for *approximate* solutions. It is not hard to prove the lemma using some theory of tensor products of normed vector spaces, as contained in [17, Chapter 1], but here is an elementary proof.

Proof. As the statement only concerns the range of A, we may assume that $A : K^s \to K^r$ is injective. In particular, we have $r \geq s$, and we prove the lemma by induction on r. For $r = s$ the matrix A is invertible, so even an exact solution to $Az = b$ exists over K. Now suppose that the statement is true for $r - 1$, which is at

least s. Denote the rows of A by $a_1, \ldots, a_r \in (K^s)^*$. As $r > s$, there exists a linear relation $\sum_i \lambda_i a_i = 0$ where not all λ_i are 0. The existence of y in the lemma yields

$$\begin{aligned} v\left(\sum_i \lambda_i b_i\right) &= v\left(\sum_i \lambda_i (b_i - a_i y) + \sum_i \lambda_i a_i y\right) \\ &= v\left(\sum_i \lambda_i (b_i - a_i y) + 0\right) \\ &> \min_i (v(\lambda_i) + l_i). \end{aligned} \tag{2}$$

After rearranging the rows of A we may assume that the latter minimum is attained in $i = r$, and by multiplying all λ_i with $1/\lambda_r$ we may assume that $\lambda_r = 1$. By the induction hypothesis, there exists a $z \in K^s$ such that $v(a_i z - b_i) > l_i$ for all $i = 1, \ldots, r-1$. For this same z we have

$$\begin{aligned} v(a_r z - b_r) &= v\left(-\sum_{i=1}^{r-1} \lambda_i a_i z - b_r\right) \\ &= v\left(-\sum_{i=1}^{r-1} \lambda_i (a_i z - b_i) - \sum_{i=1}^{r} \lambda_i b_i\right) \\ &\geq \min\left\{ v\left(\sum_{i=1}^{r-1} \lambda_i (a_i z - b_i)\right), v\left(\sum_{i=1}^{r} \lambda_i b_i\right)\right\} \\ &> \min_{i=1,\ldots,r} (v(\lambda_i) + l_i), \end{aligned}$$

where the last inequality follows from (2) and the assumption on z. By assumption, the last minimum is attained in $i = r$, and equal to $v(1) + l_r = l_r$. □

Lemma 4.4. *Let L be an algebraically closed field which is complete with respect to a non-Archimedean valuation $v : L \to \overline{\mathbb{R}}$. Set $L^0 := \{c \in L \mid v(c) \geq 0\}$, $L^+ := \{c \in L \mid v(c) > 0\}$, and $\tilde{L} := L^0/L^+$; the natural map $L^0 \to \tilde{L}$, as well as all naturally induced maps, are denoted π. Let I be an ideal in $L[x_1, \ldots, x_n]$, set $I^0 := I \cap L^0[x_1, \ldots, x_n]$ and $\tilde{I} := \pi I^0$; the latter is an ideal in $\tilde{L}[x_1, \ldots, x_n]$.*

Then for any zero $\tilde{q} \in \tilde{L}^n$ of $\tilde{I}$ there exists a zero $q \in (L^0)^n$ of I for which $\pi(q) = \tilde{q}$.

Proof. We prove this through an excursion to affinoid algebras; all properties of these algebras that are used but not proved here can be found in [9, 17]. The motivation for such an excursion is the following: let T_n be the Tate algebra of all power series $\sum_{\alpha \in \mathbb{N}^n} c_\alpha x^\alpha$ over L for which $\lim_{(\sum_i \alpha_i) \to \infty} c_\alpha = 0$. View $L[x_1, \ldots, x_n]$ as a subalgebra of T_n and let J be the ideal in T_n generated by I. Then zeroes of I (and of J) in the polydisk $(L^0)^n$ correspond bijectively to maximal ideals of the affinoid algebra $A := T_n/J$.

To find such a zero we will use a lifting theorem from the theory of affinoid algebras, whose formulation needs some further notions. First, T_n is a Banach algebra with the Gauss norm corresponding to the ring valuation $v(\sum_\alpha c_\alpha x^\alpha) = \min_\alpha v(c_\alpha)$. As any ideal of T_n, J is closed—indeed, it is the closure of I—and the affinoid algebra $A = T_n/J$ can be given the quotient norm, turning it into a Banach algebra, as well. (In fact, any Banach algebra structure on A is equivalent to this one.) Let A^0 denote the subring of A consisting of all a for which the

sequence $(a^n)_n$ is bounded, and let A^+ denote the ideal in A^0 consisting of all a with $\lim_{n\to\infty} a^n = 0$. One can show that A^0 maps any zero q of J in $(L^0)^n$ into L^0, while A^+ maps it into L^+, so that q induces a $\tilde{L}$-algebra homomorphism $\tilde{A} := A^0/A^+ \to \tilde{L}$. We thus get a map from {zeroes of J in $(L^0)^n$} to {$\tilde{L}$-algebra homomorphisms $\tilde{A} \to \tilde{L}$}. Theorem 3.5.3(ii) in [17] says that this map is surjective; this is the lifting theorem alluded to before.

To use this theorem, we must construct a homomorphism $\tilde{A} \to \tilde{L}$ from our point $\tilde{q}$. This goes as follows: first let $q' \in (L^0)^n$ be *any* lift of $\tilde{q}$. By assumption I^0 maps q' into L^+ and hence, since $J^0 := \{j \in J \mid v(j) \geq 0\}$ is the closure of I^0 and L^+ is closed, J^0 also maps q' into L^+. We conclude that the map $\phi : T^0/J^0 \to \tilde{L}$ sending $f + J^0$ to $\pi(f(q'))$ is a well-defined ring homomorphism, which restricts to π on L^0. Since A^0 is integral over its subring T^0/J^0 [17, Theorem 3.5.3(i)(2)] we can find a prime Q in A^0 lying over the prime $\ker\phi$, so that the following diagram commutes:

$$\begin{tikzcd} T^0/J^0 \arrow[r, hook] \arrow[d, "\phi"'] & A^0 \arrow[d] \\ \tilde{L} \arrow[r, hook] & A^0/Q \end{tikzcd}$$

Now the lower horizontal inclusion is again integral, and since $\tilde{L}$ is an algebraically closed field it is an isomorphism. In other words, ϕ factorises through a ring homomorphism $A^0 \to \tilde{L}$—which on L^0 still equals π, of course. From $A^+ = L^+A^0$ and $\phi(L^+) = 0$ we conclude that A^+ is mapped to zero under this homomorphism, and we have the desired $\tilde{L}$-algebra homomorphism $\psi : \tilde{A} \to \tilde{L}$.

By the aforementioned theorem, we can now find a zero $q \in (L^0)^n$ of J such that $\pi(f(q)) = \psi(f + A^+)$ for all $f \in A^0$. Taking for f the coordinate function $x_i + J_0 \in T^0/J^0 \subseteq A^0$ we find that $\pi(q_i) = \psi(x_i + J_0 + A^+) = \phi(x_i + J_0) = \tilde{q}_i$, as desired. □

Proof of Theorem 4.2. The inclusion (1)⊆(2) follows from the fact that, for a point $p \in X(L)$, the map $K[X] \to \overline{\mathbb{R}}$, $\bar{f} \mapsto v(\bar{f}(p))$ is a ring valuation extending v. For the inclusion (2)⊆(3) let $w \in T_{\bar{x}}(X)$ and let $v' : K[X] \to \overline{\mathbb{R}}$ be a ring valuation extending v with $v'(\bar{x}_i) = w_i$. Let $f = \sum_\alpha c_\alpha x^\alpha \in I$. We show that $\mathrm{in}_w f$ is not a monomial: Indeed, suppose that it is. Then on the one hand, $v'(\bar{f}) = v'(0) = \infty$ by the first axiom for ring valuations, while on the other hand $v'(\bar{f}) = v'(\sum_\alpha c_\alpha \bar{x}^\alpha)$. Now the value $v'(c_\alpha \bar{x}^\alpha)$ equals $\mathrm{wt}_w c_\alpha x^\alpha$ by definition of wt and the choice of v', and by assumption a unique term of f minimises this value. The axioms of ring valuations readily imply that $v'(\bar{g} + \bar{h}) = v'(\bar{g})$ whenever $v'(\bar{g}) < v'(\bar{h})$, hence $v'(\bar{f})$ equals that uniquely assumed minimal value, which is smaller than ∞—a contradiction to $v'(\bar{f}) = \infty$.

For the inclusion (3)⊆(4), let $w \in \overline{\mathbb{R}}^n$ be such that $\mathrm{in}_w f$ is not a monomial for any $f \in I$ and suppose that $\mathrm{in}_w \sum_{j=1}^s c_j f_j$ *is* a monomial g for some $c_j \in L$ and $f_j \in I$. Let $g_1 := g, g_2, \ldots, g_r$ be the collection of all monomials occurring in the f_i, and set $l_i := \mathrm{wt}_w(g) - \mathrm{wt}_w(g_i)$ for $i = 1, \ldots, r$. Let A be the $r \times s$-matrix over K whose entry at position (i, j) is the coefficient of g_i in f_j. Then the vector $c := (c_1, \ldots, c_s)^t \in L^s$ satisfies $v((Ac - e_1)_i) > l_i$ for all $i = 1, \ldots, r$; hence by Lemma 4.3 there also exists a $c' := (c'_1, \ldots, c'_s)^t \in K^s$ for which $v((Ac' - e_1)_i) > l_i$ for all i. But then $\mathrm{in}_w \sum_i c'_i f_i \in I$ is a non-zero scalar multiple of $g = g_1$, as well—a contradiction to (3).

Finally, for (4)$\subseteq$(1), let $w \in \overline{\mathbb{R}}^n$ be such that $\mathrm{in}_w f$ is not a monomial for any $f \in L \otimes_K I$. We have to exhibit a point $p = (p_1, \ldots, p_n) \in X(L) \subseteq L^n$ with $v(p_i) = w_i$ for all i. Choose $t_1, \ldots, t_n \in L$ with $v(t_i) = w_i$ for all i, set $x' := (x_i)_{i:w_i \neq \infty}$, and let $\phi : L[x] \to L[x']$ be the epimorphism sending f to $f(t_1 x_1, \ldots, t_n x_n)$; $\phi(f)$ only contains variables x_i with $w_i \neq \infty$ since the remaining ones have $t_i = 0$. We then have $\mathrm{wt}_w f = \mathrm{wt}_0 \phi(f)$ and $\phi(\mathrm{in}_w f) = \mathrm{in}_0 \phi(f)$. Let $J \subseteq L[x']$ be the ideal $\phi(L \otimes_K I)$; by construction $\mathrm{in}_0 f$ is not a monomial for any $f \in J$. We claim that there exists a $q \in L^{\{i:w_i \neq \infty\}}$ such that $v(q_i) = 0$ for all i and such that J vanishes on q; then setting $p_i := t_i q_i$ if $w_i \neq \infty$ and $p_i := 0$ if $w_i = \infty$ gives a point p as required.

Now retain the notation $L^0, L^+, \tilde{L}, \pi$ from Lemma 4.4. Let $J^0 := J \cap L^0[x']$, and and set $\tilde{J} := \pi J^0$. Since for any $f \in J^0$ either $\pi(f)$ is 0 or $\pi(f)$ has the same monomials as $\mathrm{in}_0 f$, the ideal $\tilde{J} \subseteq \tilde{L}[x']$ contains no monomials. As $\tilde{L}$ is algebraically closed, $\tilde{J}$ has a zero $\tilde{q}$ in $(\tilde{L}^*)^{\{i:w_i \neq \infty\}}$ by the Nullstellensatz. Applying Lemma 4.4 to $J \subseteq L[x']$, we conclude that $\tilde{q}$ can be lifted to a zero $q \in (L^0)^{\{i:w_i \neq \infty\}}$ of J. Clearly all components of q have valuation 0, so we are done.

□

By Theorem 4.2, $T_{\bar{x}}(X)$ is the intersection of infinitely many polyhedral sets, one for each element f of I: the set of all $w \in \overline{\mathbb{R}}^n$ for which $\mathrm{in}_w f$ is not a monomial. One can show that, in fact, finitely many of these polyhedral sets already cut out $T_{\bar{x}}(X)$ [27], so that the latter set is a polyhedral complex. The following theorem, originally due to Bieri and Groves [7] and also proved in [27] using Gröbner basis methods, relates the dimension of this polyhedral set to that of X.

Theorem 4.5. *Suppose that X is irreducible and of dimension d. Then $T_{\bar{x}}(X)$ is a polyhedral complex in $\overline{\mathbb{R}}^n$ which is pure of dimension d.*

As mentioned before, there is no obvious tropicalisation of morphisms between embedded affine varieties. However, *polynomial maps* do have natural tropicalisations.

Definition 4.6. For a polynomial $h \in K[x_1, \ldots, x_m]$, the map

$$T(h) : \overline{\mathbb{R}}^m \to \overline{\mathbb{R}}, \ w \mapsto \mathrm{wt}_w h$$

is called the *tropicalisation* of h. Similarly, for a polynomial map $f = (f_1, \ldots, f_n) : K^m \to K^n$, the map

$$T(f) : \overline{\mathbb{R}}^m \to \overline{\mathbb{R}}^n, \ T(f) := (T(f_1), \ldots, T(f_n))$$

is called the *tropicalisation* of f.

Note that $T(f)$ is continuous when we give $\overline{\mathbb{R}}$ the usual topology of a half-open interval. The following lemma is also well-known; see for instance [24, Theorem 3.42] for a more detailed statement. I include its short proof for self-containedness.

Lemma 4.7. *Let $f : K^m \to K^n$ be a polynomial map, let X be the Zariski closure of $\mathrm{im}(f)$, and let $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)$ be the restrictions to X of the standard coordinates on K^n. Then $T(f)$ maps $\overline{\mathbb{R}}^m$ into $T_{\bar{x}}(X)$.*

Proof. First let $w \in \mathbb{R}^n$ be such that for all $i = 1, \ldots, n$ either f_i is identically zero or $\mathrm{in}_w f_i$ is a monomial; note that then $T(f)$ is affine-linear near w (disregarding the infinite entries corresponding to the f_i that are identically 0). Choose $p =$

$(p_1, \ldots, p_m) \in L^m$ with $v(p_i) = w_i$ for all i. Then the definition of $T(f_i)$ implies that $v(f_i(p)) = T(f_i)(w)$ for all i. Hence $T(f)$ maps such a w into $T_{\bar{x}}(X)$ by the inclusion (1) $\subseteq$ (2) of Theorem 4.2. The set of all w with the required properties is clearly dense in $\overline{\mathbb{R}}^n$, $T(f)$ is continuous, and $T_{\bar{x}}(X)$ is closed—whence the lemma. $\square$

5. Proofs of Theorem 2.1 and Corollary 2.3

We retain the notation of Section 2.

Proof of Theorem 2.1. Consider the map $f : \prod_{i=1}^k K^{m_i} \to C_1 + \ldots + C_k \subseteq V$ sending $(p_1, \ldots, p_k)$ to $f_1(p_1) + \ldots + f_k(p_k)$. Endow K with the trivial valuation, and observe that the map $\prod_{i=1}^k \mathbb{R}^{m_i} \to \mathbb{R}^m$ whose b-th component is $\min_{i=1,\ldots,k} l_{i,b}$ is precisely the tropicalisation $T(f)$ of f—or rather, its restriction to the set $\prod_{i=1}^k \mathbb{R}^{m_i}$. Hence the set Q in the theorem is precisely $T(f)(\prod_{i=1}^k \mathbb{R}^{m_i})$. By Lemma 4.7 this set is contained in the tropicalisation $T_{\bar{y}}(C_1 + \ldots + C_k)$ (where $y = (y_1, \ldots, y_n)$ are the standard coordinates on K^n), and hence by Theorem 4.5 its dimension does not exceed the dimension of $C_1 + \ldots + C_k$. $\square$

Proof of Corollary 2.3. Let f be the polynomial map from the previous proof, so that $T(f)$ is a piecewise linear map $\prod_{i=1}^k \mathbb{R}^{m_i} \to \mathbb{R}^n$. Let $v \in \prod_i \mathbb{R}^{m_i}$ and let $B \subseteq \{1, \ldots, n\}$ be the set of indices won at v by some $i \in \{1, \ldots, k\}$; that is, we simply leave out the indices where there is a tie. Then the map F, defined as the composition of $T(f) : \prod_i \mathbb{R}^{m_i} \to \mathbb{R}^n$ and the projection $\mathbb{R}^n \to \mathbb{R}^B$, is linear near v, so that its differential $d_v F : \prod_i \mathbb{R}^{m_i} \to \mathbb{R}^B$ is well-defined. Indeed, for $i = 1, \ldots, k$ and $b \in B$ the b-th component of the restriction of $d_v F$ to $\mathbb{R}^{m_i}$ is $d_{v_i} l_{i,b}$ if i wins b, and 0 otherwise. Hence the rank of $d_v T(f)$ is exactly

$$\sum_{i=1}^k \dim_{\mathbb{R}} \langle D_i(v) \rangle_{\mathbb{R}};$$

therefore this number is a lower bound on $\dim \operatorname{im} F$, which in turn is a lower bound on $\dim \operatorname{im} T(f)$, and hence to $\dim(C_1 + \ldots + C_k)$ by the proof of Theorem 2.1. $\square$

Note that the dimension of Q in Theorem 2.1 is in fact equal to the maximum rand of $d_v T(f)$ over all a points v where this rank is linear.

6. Minimal orbits

An important class of varieties, whose non-defectiveness is notoriously hard to prove, are the *minimal orbits*. To define these, let G be a connected reductive algebraic group over K, and let V be an irreducible G-module. Then $\mathbb{P}V$ has a unique closed G-orbit (see for instance [8, 19, 20] for general theory of algebraic groups, Lie algebras, and their representations), the cone over which is $C := Gv_\lambda \cup \{0\} \subseteq V$, where v_λ is the highest weight vector of V relative to some Borel subgroup B. Many interesting cones C arise in this manner.

Example 6.1. (1) Let G be GL_m, let $d \in \{1, \ldots, m-1\}$, and set V equal to $\bigwedge^d (K^m)$. Then C is the cone over the Grassmannian, in its Plücker embedding, of d-dimensional subspaces of K^m.

(2) Let G be $(\mathrm{GL}_m)^d$, let $d \in \mathbb{N}$, and let V be the space $(K^m)^{\otimes d}$. Then C is the cone of all *pure d-tensors*, i.e., those that can be written as $u_1 \otimes \ldots \otimes u_d$ for some $u_1, \ldots, u_d \in K^m$; it is the cone over the Segre embedding of the d-fold Cartesian power of $\mathbb{P}^{m-1}$.

(3) Let G be GL_m, let $d \in \mathbb{N}$, and let V be the space of homogeneous polynomials of degree d on K^m. Now C is the cone of all pure d-th powers of linear forms, and the affine cone over the Veronese embedding of $\mathbb{P}^{m-1}$ of degree d. It is well known that the dimension of kC is the codimension of the space of all homogeneous polynomials f of degree d for which both f and all first order partial derivatives of f vanish in k generic points on $\mathbb{P}^{m-1}$. This relates the secant dimensions of C to *multivariate polynomial interpolation*. These dimensions are known for all m and d from the papers [1, 2, 3, 18].

(4) Let G be SO_m, and let V be the Lie algebra of G. Then C is the cone over the Grassmannian, in its Plücker embedding, of isotropic 2-dimensional subspaces of K^m. The secant varieties of C were determined in [5].

There is a general argument showing that the first few secant varieties of minimal orbits are non-defective. To state it, let T be a maximal torus in B, let $P \supseteq B$ be the stabiliser of Kv_λ, and let U be the unipotent radical of the parabolic subgroup opposite to P containing T. Then $\mathfrak{g}v_\lambda = \mathfrak{u}v_\lambda \oplus Kv_\lambda$. Finally, let W be the Weyl group of (G, T); for $w \in W$ let $\tilde{w}$ be an element of $N_G(T)$ representing w.

Proposition 6.2. *Let $k \in \mathbb{N}$ and $w_1, \ldots, w_k$ be elements of W. Then*

$$\dim kC \geq \dim \sum_{i=1}^{k} \tilde{w}_i(\mathfrak{u}v_\lambda + Kv_\lambda).$$

In particular, if there exist $w_1, \ldots, w_k$ for which the spaces $\tilde{w}_i(\mathfrak{u}v_\lambda + Kv_\lambda)$ are linearly independent, then kC is non-defective.

Proof. The rank of the differential of the addition map $C^k \to kC$ at any point of C^k is a lower bound on the dimension of kC, which in an open dense subset of C^k is exact—this is Terracini's lemma [29]. Now take for this point the point $(\tilde{w}_1 v_\lambda, \ldots, \tilde{w}_k v_\lambda)$. The tangent space to C at $\tilde{w}_i v_\lambda$ is $\tilde{w}_i \mathfrak{g} v_\lambda = \tilde{w}_i(\mathfrak{u}v_\lambda + Kv_\lambda)$, and the differential of the summation map maps the tuple of these spaces to their sum. □

Proposition 6.2 is useful for small k and large highest weights: the space $\mathfrak{u}v_\lambda + Kv_\lambda$ is contained in the sum of the weight spaces with weights $\lambda - \alpha$, where λ is the weight of v_λ and $-\alpha$ is 0 or a root whose root space lies in $\mathfrak{u}$. Hence if there exist $w_1, \ldots, w_k$ such that the translates

$$w_i\{\lambda - \alpha \mid -\alpha \text{ is a root of } \mathfrak{u}\}$$

are all disjoint, then kC is non-defective. For λ large and deep in the interior of the dominant chamber, there will exist such w_i for all k up to $|W|$. However, for $k > |W|$, the bound of the proposition is evidently off.

The bound from Proposition 6.2 is also quite good for representations that are small in the sense that all weights, or many of them, are highest weights. Let me illustrate this in the first three examples of 6.1. For the coding theory notions appearing here and in what follows, I refer to [21].

Example 6.3. **Grassmannians:** Take for T the diagonal matrices. The W-orbit of the highest weight vector consists of the vectors $e_{i_1} \wedge \ldots \wedge e_{i_d}$ with $1 \leq i_1 < \ldots < i_d \leq m$. These vectors correspond naturally to the binary words in $\{0,1\}^m$ of Hamming weight d. Given a collection B of k such words, the lower bound of the proposition equals the number of weight-d, length-m binary words at distance at most 2 from B. Hence, if there exists a binary code B of size k, length m, constant weight d, and minimal Hamming distance distance 6, then kC is non-defective. Variants of this idea already appeared in [15, 12].

Pure tensors: Take for T the d-tuples of diagonal matrices. The W-orbit of the highest weight vector consists of all tensors of the form $e_{i_1} \otimes \ldots \otimes e_{i_d}$, where $i_1, \ldots, i_d \in \{1, \ldots, m\}$. These correspond naturally to the m-ary words of length d. Given a collection B of such words, the lower bound from the proposition equals the number of words at Hamming distance at most 1 from B, or, equivalently, to the total number of fields on a d-dimensional chessboard with side lengths m covered by rooks on the positions in B. Hence, if there exists an m-ary code B of size k, length d, and minimal Hamming distance 3, then kC is non-defective. The existence of perfect (Hamming) codes for $m = p^e$ and $d = (p^{ef} - 1)/(p^e - 1)$, where p is a prime and e and f are arbitrary positive natural numbers, shows that the corresponding C are non-defective. This idea is contained in [10, 15, 28].

Homogeneous polynomials: Take for T the diagonal matrices. Here the weight vectors are the monomials, corresponding naturally to the multi-indices $\alpha \in \mathbb{N}^m$ with $|\alpha| := \sum_{i=1}^m \alpha_i = d$. The elements of the form de_i correspond to the elements x_i^d in the W-orbit of the highest weight vector. If B is a set of such de_i, then the lower bound from the proposition is the number of α that are at 1-distance at most 2 from B; this is also mentioned in [15]. Dually, if B has size k and consists of elements that are mutually at 1-distance at least 6, then kC is non-defective. Note that such a B exists if and only if $k \leq m$ and $d \geq 3$.

For $d = 2$ the bound turns out to be exact: Taking $B = \{2e_1, \ldots, 2e_k\}$, where $k \leq m$, the proposition says that kC has dimension at least $m + (m-1) + \ldots + (m-k+1)$, and this is known to be the exact dimension.

It would be very interesting to apply Corollary 3.2 to minimal orbits in general irreducible representations, but for this one needs a suitable basis of the representation V, and it is not obvious which one to use. This will be the theme of further research; here we content ourselves with the following lower bound, which seems to be good for Segre products of Veronese embeddings. Write $X(V)$ for the set of T-weights on V, considered as a subset of the real vector space $\mathbb{R} \otimes_{\mathbb{Z}} X(T)$ spanned by the character group of T.

Proposition 6.4. *For any positive definite inner product on* $\mathbb{R} \otimes_{\mathbb{Z}} X(T)$ *and any* k*, the optimum of* $\textsc{VoronoiPartition}(X(V), k)$ *is a lower bound on* $\dim kC$.

Proof. To apply Corollary 3.2 we need a dominant polynomial map into C together with choices of basis. For the map we take

$$f : K \times \mathfrak{u} \to C \subseteq V, \ (t, u) \mapsto t \exp(u) v_\lambda.$$

Let $X(\mathfrak{u})$ denote the set of T-roots in $\mathfrak{u}$ and set $\tilde{X}(\mathfrak{u}) := \{0\} \cup X(\mathfrak{u})$. In $K \times \mathfrak{u}$ we choose a basis labelled by $\tilde{X}(\mathfrak{u})$, where 0 corresponds to $(1, 0)$ and $\beta \in X(\mathfrak{u})$ corresponds to a root vector u_β in $\mathfrak{u}$ with root β. Let $(x_\beta)_{\beta \in \tilde{X}(\mathfrak{u})}$ be the corresponding coordinates on $K \times \mathfrak{u}$. Finally, in V we choose any basis $v_1, \ldots, v_n$ of T-weight vectors.

For $b = 1, \ldots, n$ let A_b be the set of exponent vectors of monomials in the $x_\beta, \beta \in \tilde{X}(\mathfrak{u})$, occurring in f_b. Observe that if μ is the weight of v_b, then these exponent vectors $(r_\beta)_{\beta \in \tilde{X}(\mathfrak{u})}$ all satisfy $r_0 = 1$ and furthermore $\lambda + \sum_\beta r_\beta \beta = \mu$—only such monomials in the u_β can map v_λ to an element having non-zero b-th component. By the latter equality, Corollary 3.2, and Lemma 3.8, $\textsc{AffinePartition}((A_b)_b, k)$ is a lower bound on $\dim kC$.

Now let π be the affine-linear map from $(\mathbb{R}^{\tilde{X}(\mathfrak{u})})^*$ to $\mathbb{R} \otimes_{\mathbb{Z}} X(T)$ sending $(r_\beta)_\beta$ to $\lambda + \sum_\beta r_\beta \beta$; the above shows that $\pi(A_b) = \{\mu\}$ if v_b has weight $\mu \in X(V)$. Hence, by Remark 3.6 and Lemma 3.8, $\textsc{AffinePartition}((\{\mu\})_\mu, k)$ and $\textsc{VoronoiPartition}(X(V), k)$ are lower bounds to $\dim kC$, as well. □

Clearly, if V_μ is 1-dimensional for all μ, then the bound of Proposition 6.4 equals $\dim V$ for k sufficiently large—a property that did not hold for the bound of Proposition 6.2. This shows that the bound is not *useless*, as in Example 3.3. I do not claim that the bound of Proposition 6.4 is *good* for all weight-1 representations. For instance, for C the cone over the Grassmannian of d-dimensional subspaces of an m-dimensional subspace, the bound only gives $\dim C \geq m$ rather than $d(m - d) + 1$. This is due to the rather coarse reduction using π; so to apply Corollary 3.2 to Grassmannians one needs a more subtle approach. When $X(\mathfrak{u})$ is linearly independent, however, as is the case for Veronese and Segre embeddings and indeed for Segre products of Veronese embeddings, then π is an isomorphism, and the bound of Proposition 6.4 seems actually very good, as we will see in the next section.

There is much more to be said here: what about representations where the weight multiplicities are not 1? What about representations where they are, but $X(\mathfrak{u})$ is linearly dependent? Such questions will be addressed in forthcoming research; I conclude this paper with a few concrete examples of how strong the lower bounds from the tropical approach can be.

7. Applications

7.1. **Veronese embeddings.** Let m, d be natural numbers and set $V := \mathbb{C}[t]_d$, the space of complex homogeneous polynomials of degree d in the variables $t = (t_1, \ldots, t_m)$. As a basis of V we choose the monomials t^α, where α runs through the set $X \subseteq \mathbb{N}^m$ of all multi-indices with $|\alpha| = d$. Let $f : K^m \to V$ send $(x_1, \ldots, x_m)$ to the pure power $(x_1 t_1 + \ldots + x_m t_m)^d$, and let C be the image of f. The coefficient of t^α in $f(x)$ is a non-zero scalar times x^α, so—like in Example 3.4, where m was 2—every A_α consists of the single vector α. So Corollary 3.2 leads us to consider the optimisation problem $\textsc{LinearPartition}(\{\{\alpha\}\}_{\alpha \in X}, k)$ to bound the dimension of kC. As X lies on an affine hyperplane not through 0, we may by Lemma 3.8 just as well solve $\textsc{AffinePartition}(\{\{\alpha\}\}_{\alpha \in X}, k)$, and since all sets A_b are singletons, also $\textsc{VoronoiPartition}(X, k)$ gives a lower bound on $\dim kC$; this is also the content of Proposition 6.4 for the representation of GL_m on V.

Of course we know the dimension of kC already from [3], but it would be very desirable to have an alternative, more elementary proof of their results. I think

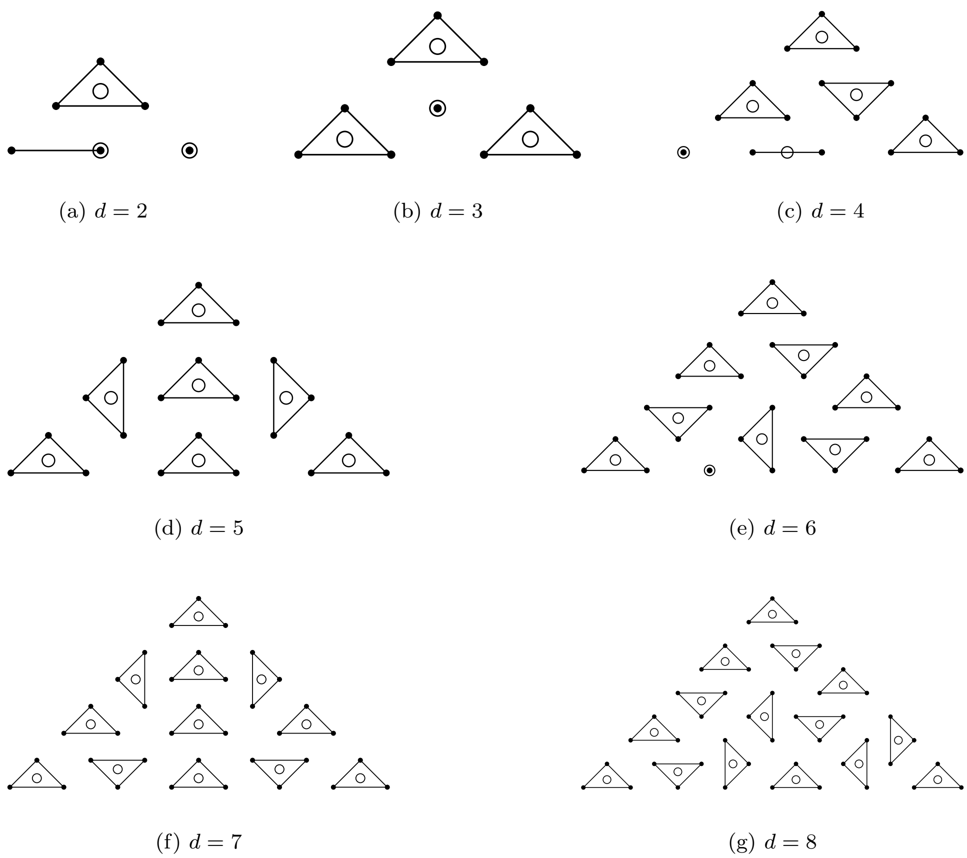
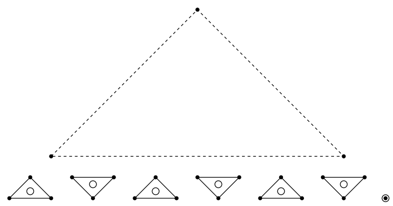


FIGURE 3. Non-defective figures exist for $d \neq 2, 4$.

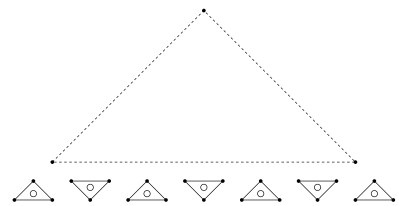
that the tropical approach might yield such a proof. To motivate this belief, let us prove the first non-trivial case solved by Hirschowitz in [18], namely, the case where $m = 3$. Rick Miranda and Olivia Dumitrescu have also proved the following theorem, using degenerations (private communication). The combinatorics to which their proof boils down resembles very much the combinatorics below, and it would be interesting to understand exactly how both approaches are connected.

Theorem 7.1. *Suppose that $m = 3$. Then kC has the expected dimension unless $(d, k) = (2, 2)$ or $(4, 5)$, in which cases the defect $k \dim C - \dim kC$ equals 1.*

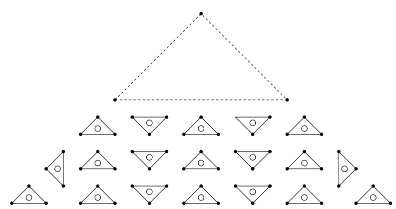
Proof. That C is defective for $d = 2, 4$, is well known (for $d = 4$ by the work of Clebsch [13]); in those two cases we will only show that the defect is not more than 1. We give a pictorial proof of the theorem: the elements of X form a triangle in the plane in $\mathbb{R}^3$ where the sum of the coordinates equals d. If we draw points $v_1, \ldots, v_k$ in that triangle, and if the points of X lying in the Voronoi cell of v_i span an affine space of dimension $d_i \in \{0, 1, 2\}$, then $\sum_i (d_i + 1)$ is a lower bound on $\dim kC$ by Corollary 3.2 and Lemma 3.8. Moreover, we may and choose any 2-norm on the plane containing X. We choose to draw X as in Figure 3, and choose the 2-norm



(a) From $d-2$ to $d \equiv 0 \mod 3$



(b) From $d-2$ to $d \equiv 1 \mod 3$



(c) From $d-6$ to $d \equiv 2 \mod 3$

FIGURE 4. Inductive construction of non-defective figures

for which circles really look like circles in the plane. In this manner, Figure 3(a) gives lower bounds $3, 5, 6$ for the dimensions of $C, 2C, 3C$ when $d = 2$: take for the v_i the midpoints of the open circles, and group the (black) points of X according to closest v_i; this results in a triangle, an edge, and a single point.

We call such a picture non-defective if the groups are only triangles, together with a single point if $d \equiv 0 \mod 3$; then the picture shows that the corresponding C is non-defective. Figures 3(b)–3(g) prove the theorem for $d = 3, \ldots, 8$; all except Figure 3(c) are non-defective. Now we proceed by induction: we can produce a non-defective picture for $d \geq 9,\ d \equiv 0 \mod 3$ from a non-defective picture for $d-2$ as indicated in Figure 4(a). Similarly, we construct a non-defective picture for $d \geq 7,\ d \equiv 1 \mod 3$ from a non-defective picture for $d-2$ as in Figure 4(b). Finally, for $d \geq 11,\ d \equiv 2 \mod 3$, we construct a picture from a non-defective picture for $d-6$ as in Figure 4(c). One readily verifies that this yields non-defective figures for all $d \geq 8$ and hence proves the theorem. $\square$

As promised, we compare the bound of Proposition 6.2 to that of Corollary 3.2.

Proposition 7.2. *The optimum of* VORONOIPARTITION(X, k)*, relative to the standard inner product, is at least that of Proposition 6.2 applied to Veronese embeddings.*

Proof. Recall the bound of Proposition 6.2 for the cone of pure powers: for indices $1 \leq i_1 < \ldots < i_k \leq n$ the lower bound on $\dim kC$ in that proposition is the set of all words at 1-distance at most 2 from the set $\{de_{i_1}, \ldots, de_{i_k}\} =: B$. First suppose that $d \geq 3$, so that the lower bound equals km. Choose $v_i := de_{i_j}$, set $v := (v_1, \ldots, v_k) \in (\mathbb{R}^m)^k$, and consider the Voronoi diagram of the v_i. Then the 2-distance between distinct v_j is $d\sqrt{2} > 2\sqrt{2}$, so that the Voronoi cell of v_j contains

all $\alpha \in X$ at 2-distance at most $\sqrt{2}$ from v_j. But these are exactly the m vectors in X at 1-distance at most 2 from de_{i_j}, and their affine span $(m-1)$-dimensional; this shows that $\sum_i (1 + \dim \mathrm{Aff}_{\mathbb{R}} D_i(v))$ is indeed at least the bound of Proposition 6.2.

Now suppose that $d = 2$. First choose the v_j as above. Then de_{i_j} still has v_j as the unique closest point among all v_l, but v_j and v_l are equally close to $\alpha = e_{i_j} + e_{i_l}$. Now perturb all v_js slightly to $v'_j := v_j + \epsilon_j$ to resolve these ties, such that relative to the Voronoi diagram of the v'_j the vector $2e_{i_j}$ still lies in the cell of v'_j, while $e_{i_j} + e_{i_l}$ either lies in the cell of v'_j or in that of v'_l, for all $j \neq l$. Then, writing $v' := (v'_j)_j$, each of the α at 2-distance $\sqrt{2}$ from B contributes exactly 1 to the sum

$$\sum_{i=1}^{k} (1 + \dim \mathrm{Aff}_{\mathbb{R}} D_i(v')),$$

and therefore this sum is equal to $n + (n-1) + \ldots + (n-k+1)$, just like the (exact) bound of Proposition 6.2. □

The proof of Theorem 7.1 shows that the optimum of VoronoiPartition is in fact much stronger than that of Proposition 6.2: while the latter bound only takes corners of X into account, the former bound also exploits the interior of X. We will encounter a similar phenomenon with Segre varieties.

7.2. **Segre varieties.** Let m, d be natural numbers and set $V := (K^m)^{\otimes d}$. Let X be the set of $d \times m$-matrices of natural numbers whose row sums are all 1. For $\alpha \in X$ we set $e_\alpha := e_{j_1} \otimes \ldots \otimes e_{j_d}$, where j_i is the column containing the 1 in the i-th row; the e_α form our basis of V. Let C be the closed cone of of pure tensors in V. Writing $M_{d,m}(K)$ for the space of $d \times m$-matrices with entries in K, we parametrise C by the polynomial map $M_{d,m}(K) \to V$ sending x to

$$(x_{11}e_1 + \ldots + x_{1m}e_m) \otimes \ldots \otimes (x_{d1}e_1 + \ldots + x_{dm}e_m) = \sum_{\alpha \in X} x^\alpha e_\alpha.$$

By Corollary 3.2, the optimum of LinearPartition$(\{\{\alpha\}\}_{\alpha \in X}, k)$ is a lower bound on $\dim kC$ for all k. Moreover, X is contained in the affine hyperspace of $M_{d,m}(\mathbb{R})$ where all row sums are 1, so that also the optima of AffinePartition$(\{\{\alpha\}\}_{\alpha \in X}, k)$ and VoronoiPartition(X, k) are lower bounds on $\dim kC$ by Lemma 3.8. This last statement is the content of Proposition 6.4 for the minimal orbit of the representation of GL_m^d on V. First we show that the last lower bound is at least as good as that of Proposition 6.2.

Proposition 7.3. *The optimum of* VoronoiPartition(X, k)*, relative to the standard inner product on* $M_{d,r}(\mathbb{R})$*, is at least the lower bound of Proposition 6.2 applied to Segre powers.*

Proof. The proof is very similar to that of Proposition 7.2: the bijection ϕ from $\{1, \ldots, m\}^d$ to X sending $(j_1, \ldots, j_d)$ to the matrix having 1s on the positions (i, j_i) and 0s elsewhere has the property that $||\phi(w_1) - \phi(w_2)||_2 = \sqrt{2b}$ if b is the Hamming distance between w_1 and w_2. Let $w_1, \ldots, w_k$ be distinct m-ary words of length d, set $v_i := \phi(w_i)$ and $v := (v_1, \ldots, v_k)$. The lower bound on $\dim kC$ of Proposition 6.2 is then size of the set

$$S := \{\alpha \in X \mid ||\alpha - v_i|| \leq \sqrt{2} \text{ for some } i\}.$$

Note that for every i the elements of S at 2-distance $\leq \sqrt{2}$ from v_i form an affinely independent set: they are either equal to v_i or obtained from v_i by moving a 1 within its row. Some elements of S may have distance $\sqrt{2}$ to two distinct v_i. Perturbing v slightly to resolve these ties yields a v' such that $S \subseteq \bigcup_i D_i(v')$ and every $S \cap D_i(v')$ is contained in the closed 2-ball of radius $\sqrt{2}$ from v_i—hence affinely independent. Hence $\sum_{i=1}^{k}(1 + \dim \mathrm{Aff}_{\mathbb{R}} D_i(v'))$ is at least $|S|$, which is the lower bound from Proposition 6.2. □

The following proposition shows that for Segre powers, too, the optimum of VoronoiPartition can be strictly larger than the lower bound of Proposition 6.2.

Proposition 7.4. *The 6-th Segre power of the projective line is non-defective.*

In [11] it is proved that all higher secant varieties of Segre powers of the projective line are non-defective, except possibly for one higher secant variety of each Segre power. The statement of the proposition is the first case not covered by the theorem in [11], and what follows is the first computer-free proof of that statement.

It will be convenient to work with the set $Y \subseteq M_{d,m-1}(\mathbb{R})$ obtained from X by deleting the last column of every element of X; note that this operation defines an affine equivalence from X to Y, so that it does not affect the optimum of AffinePartition. We will work with VoronoiPartition relative to the standard inner product on $M_{d,m-1}(\mathbb{R})$; note that its restriction to Y is *not* equal to that on $M_{d,m}(\mathbb{R})$ restricted to X and transferred to Y by the affine equivalence—but in VoronoiPartition we are free to choose our inner product!

Proof. We are in the situation where $m = 2$ and $d = 6$. We have $\dim C = 7$ and $\dim V = 64$, so we have to show that $9C$ has the expected dimension 63. This is impossible using only the rook covering bound of Proposition 6.2, because the maximal size of a binary code of length 6 and Hamming distance 3 is 8 (see, e.g., [6]). However, we will use such a code, and then complement it with a further point to take care of the points outside the Hamming balls of radius 1 around its codewords.

More specifically, note that $Y = \{0, 1\}^6$. Let B be the set of all vectors v in Y with

$$Hv = 0 \mod 2, \qquad \text{where } H = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

is the *parity check matrix* of B. From the fact that the columns of H are distinct modulo 2 one readily concludes that the minimal Hamming distance between elements of B is 3. Explicitly, B consists of the rows $v_1, \ldots, v_8$ of the following matrix:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Now set $v_9 := (\frac{1}{2}, \ldots, \frac{1}{2})$ and consider the Voronoi diagram of $v = (v_1, \ldots, v_9)$. Any element w of Y has 2-distance $\sqrt{3/2}$ to v_9. If w has Hamming distance 1 to some v_i with $i < 9$, then its 2-distance to v_i is also 1, and w lies in the Voronoi cell of v_i. Otherwise, w has 2-distance at least $\sqrt{2}$ to every v_i with $i < 9$, hence w lies in the Voronoi cell of v_9. We have to check that each $D_i(v)$ spans an affine space of dimension 6. For $i < 9$ this is clear, because $D_i(v)$ is affinely equivalent to the Hamming ball of radius 1 around 0, which apart from 0 contains all standard basis vectors. Finally, $D_9(v)$ contains $64-8*7=8$ words, and it is easy to see that these are precisely the words of the form $(1,1,1,1,1,1) - v_i$ with $i < 9$: indeed, these 8 words give syndrome $(1,1,1)^t$ when multiplied from the right by H and taken modulo 2, so they are not at Hamming distance 1 from any element of B. Hence $D_i(v)$ is affinely equivalent to B, and a direct computation shows that the affine span of B is the full space $\mathbb{R}^6$. We conclude that $9C$ is, indeed, non-defective. $\square$

7.3. **Grassmannians.** Let m, d be natural numbers with $d \leq m/2$ and set $V := \bigwedge^d(K^m)$. Let X be the set of all d-subsets of $\{1, \ldots, m\}$ and for $J \in X$ with elements $i_1 < \ldots < i_d$ set $e_J := e_{i_1} \wedge \ldots \wedge e_{i_d}$; the e_J form our basis of V. Let C be the closed cone which is the image of $M_{d,m}(K)$ under the map sending x to

$$(x_{11}e_1 + \ldots + x_{1m}e_m) \wedge \ldots \wedge (x_{d1}e_1 + \ldots + x_{dm}e_m) = \sum_J \det(x_J)e_J;$$

here x_J is the $d \times d$-matrix obtained from x by taking only the columns corresponding to elements of J. The exponent vectors of the monomials appearing in the coefficient $\det(x_J)$ of e_J are precisely those matrices in $M_{d,m}(\mathbb{N})$ that are 0 outside the columns in J and whose restriction to the columns in J are permutation matrices; call this collection A_J. Then Corollary 3.2 says that the optimum of $\textsc{LinearPartition}((A_J)_J, k)$ is a lower bound on $\dim kC$. Again, all A_J lie in a common affine space not through zero, so that also $\textsc{AffinePartition}$ yields a lower bound. Since the A_J are not singletons, we cannot immediately use $\textsc{VoronoiPartition}$. In fact, one can still prove that the optimum of $\textsc{AffinePartition}((A_J)_J, k)$ is greater than the lower bound of Proposition 6.2, but it is slightly more involved. As it fits better with forthcoming work dealing with more general secant dimensions of minimal orbits, we omit it here.

8. Conclusion

The tropical approach to secant dimensions shows very promising results when tested on concrete minimal orbits, especially those in representations where all weight spaces are one-dimensional—like the Veronese, Segre, or Plücker embeddings. The approach leads to exciting combinatorial-polyhedral questions. In particular, the approach yields a nice pictorial proof of the non-defectiveness of most Veronese surfaces, one of the main results of [18]; a similar proof for general Veronese embeddings would be an attractive alternative to [3].

Still, these polyhedral-combinatorial questions are mostly open, and there is a lot of space for further research. In particular:

(1) Using Terracini's lemma, one can compute $\dim kC$ by computing the rank of the addition map $C^k \to kC$ at a *generic* point. However, for this one has to compute the rank of a large matrix (of size the dimension of the representation); see [4]. On the other hand, the tropical approach only needs the ranks of several smaller matrices (of size the dimension of C), but it only

works at a carefully selected point where the rank of the differential of a tropical polynomial map is maximal. Is there a method in between, which does work at random points but only requires ranks of small matrices?

(2) The tropical approach depends on the chosen bases: to prove anything substantial, it seems wise to choose nice bases of both the representation and the parametrising space. For minimal orbits in representations where not all weight spaces are one-dimensional, it is unclear which bases of the representation one should use. The question of whether there exist bases for which the tropical method works well is a very exciting one!

(3) Proposition 6.4 gives very interesting lower bounds for secant dimensions of minimal orbits where the root system of the negative unipotent radical is linearly independent. In particular, these bounds apply to Segre products of Veronese embeddings. I do not know of an example where the lower bound is not sharp.

References

[1] J. Alexander. Singularités imposables en position générale à une hypersurface projective. *Compos. Math.*, 68(3):305–354, 1988.

[2] J. Alexander and A. Hirschowitz. La méthode d'Horace éclatée: Application à l'interpolation en degré quatre. *Invent. Math.*, 107(3):585–602, 1992.

[3] J. Alexander and A. Hirschowitz. Polynomial interpolation in several variables. *J. Algebr. Geom.*, 4(2):201–222, 1995.

[4] Karin Baur, Willem de Graaf, and Jan Draisma. Secant dimensions: computations and conjectures. 2006. Preprint.

[5] Karin Baur and Jan Draisma. Higher secant varieties of the minimal adjoint orbit. *J. Algebra*, 280:743–761, 2004.

[6] M.R. Best, A.E. Brouwer, F.Jessie MacWilliams, Andrew M. Odlyzko, and Neil J.A. Sloane. Bounds for binary codes of length less than 25. *IEEE Trans. Inf. Theory*, 24:81–93, 1978.

[7] Robert Bieri and J.R.J. Groves. The geometry of the set of characters induced by valuations. *J. Reine Angew. Math.*, 347:168–195, 1984.

[8] Armand Borel. *Linear Algebraic Groups*. Springer-Verlag, New York, 1991.

[9] S. Bosch, U. Güntzer, and R. Remmert. *Non-Archimedean Analysis*, volume 261 of *Grundlehren der mathematischen Wissenschaften*. Springer-Verlage, Berlin, Heidelberg, 2004.

[10] M.V. Catalisano, A.V. Geramita, and A. Gimigliano. On the rank of tensors, via secant varieties and fat points. In *Geramita, A. V. (ed.), Zero-dimensional schemes and applications. Proceedings of the workshop, Naples, Italy, February 9–12, 2000. Kingston: Queen's University.*, volume 123 of *Queen's Pap. Pure Appl. Math.*, pages 135–147, 2002.

[11] M.V. Catalisano, A.V. Geramita, and A. Gimigliano. Higher secant varieties of the Segre varieties $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$. *J. Pure Appl. Algebra*, 201(1–3):367–380, 2005.

[12] M.V. Catalisano, A.V. Geramita, and A. Gimigliano. Secant varieties of grassmann varieties. *Proc. Am. Math. Soc.*, 133(3):633–642, 2005.

[13] A. Clebsch. Über Kurven vierter Ordnung. *J. Reine Angew. Math.*, 59:125–145, 1861.

[14] Mike Develin. Tropical secant varieties of linear spaces. *Discrete Comput. Geom.*, 35(1):117–129, 2006.

[15] R. Ehrenborg. On apolarity and generic canonical forms. *J. of Algebra*, 213:167–194, 1999.

[16] Manfred Einsiedler, Mikhail Kapranov, and Douglas Lind. Non-archimedean amoebas and tropical varieties. 2004. Preprint, available from `http://arxiv.org/abs/math.AG/0408311`.

[17] Jean Fresnel and Marius van der Put. *Rigid analytic geometry and its applications*, volume 218 of *Progress in Mathematics*. Birkhäuser, Boston, 2004.

[18] André Hirschowitz. La methode d'Horace pour l'interpolation à plusieurs variables. *Manuscr. Math.*, 50:337–388, 1985.

[19] James E. Humphreys. *Introduction to Lie algebras and representation theory*. Springer, Berlin, 1972.

[20] James E. Humphreys. *Linear algebraic groups*, volume 21 of *Graduate Texts in Mathematics*. Springer-Verlag, New York - Heidelberg - Berlin, 1981.

[21] J.H. van Lint. *Introduction to coding theory*, volume 86 of *Graduate Texts in Mathematics*. Springer, Berlin, 1991.

[22] Grigory Mikhalkin. Amoebas of algebraic varieties and tropical geometry. In Simon Donaldson *et al.*, editor, *Different faces of geometry*, volume 3 of *International Mathematical Series*, pages 257–300, New York, 2004. Kluwer Academic/Plenum Publishers.

[23] Grigory Mikhalkin. Tropical geometry and its applications. 2006. Preprint, available from `http://www.arxiv.org/abs/math.AG/0601041`.

[24] Lior Pachter and Bernd Sturmfels, editors. *Algebraic statistics for computational biology*, Cambridge, 2005. Cambridge University Press.

[25] Jürgen Richter-Gebert, Bernd Sturmfels, and Thorsten Theobald. First steps in tropical geometry. In G. L. *et al.* Litvinov, editor, *Idempotent mathematics and mathematical physics. Proceedings of the international workshop, Vienna, Austria, February 3-10, 2003*, volume 377 of *Contemporary Mathematics*, pages 289–317, Providence, RI, 2005. AMS.

[26] David Speyer and Bernd Sturmfels. The tropical grassmannian. *Adv. Geom.*, 4(3):389–411, 2004.

[27] Bernd Sturmfels. *Solving systems of polynomial equations*, volume 97 of *CBMS Regional Conference Series in Mathematics*. AMS, Providence, RI, 2002.

[28] Bernd Sturmfels and Seth Sullivant. Combinatorial secant varieties. *Quart. J. Pure Applied Math.*, 2005. To appear; preprint available from `http://arxiv.org/abs/math.AC/0506223`.

[29] A. Terracini. Sulle V_k per cui la variet degli $S_h(h+1)$-seganti ha dimensione minore dell' ordinario. *Palermo Rend.*, 31:392–396, 1911.

[30] F.L. Zak. *Tangents and secants of algebraic varieties.*, volume 127 of *Translations of Mathematical Monographs*. American Mathematical Society (AMS), Providence, RI, 1993.

Department of Mathematics and Computer Science, Technische Universiteit Eindhoven, P.O. Box 513, 5600 MB Eindhoven, The Netherlands

E-mail address: `j.draisma@tue.nl`